Synthesis of Passive *RC* Networks with Gains Greater than Unity*

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Summary—**This paper illustrates the fact that three-terminal passive resistor-capacitor networks are capable of having unity voltage gain at zero frequency and higher than unity gain over a prescribed frequency band. A general design procedure for synthesizing a transfer function with these properties using, insofar as possible, known methods of synthesis is outlined. Together with suggestions for further work, two examples are presented to illustrate the method.**

I. Introduction

In THE SYNTHESIS of three-terminal passive resistor-capacitor networks to obtain a specified transfer function, the voltage gain at zero frequency is never greater than unity. It is possible, however, to synthesize three-terminal passive resistor-capacitor networks with unity voltage gain at zero frequency and higher than unity gain over some pass band.[1] A typical asymptotic plot[2] of the transfer function versus frequency obtainable with such networks is shown in Fig. 1. The graph plots the following function on logarithmic

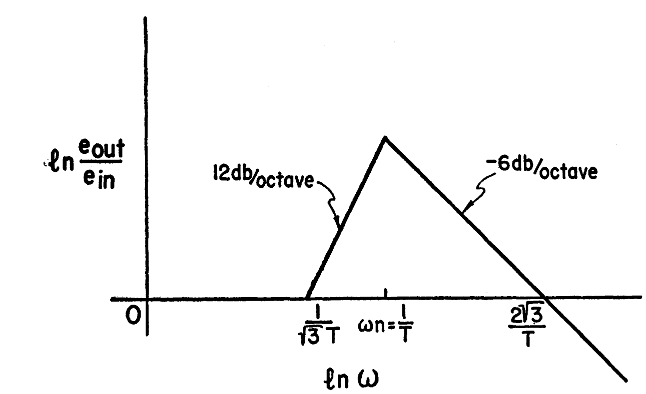

Fig. 1—Asymptotic logarithmic plot of a transfer function.

scales, assuming that no section of the network loads the preceding sections:

$$\frac{e_{\text{out}}}{e_{\text{in}}} = \frac{3T^2s^2 + 3Ts + 1}{(Ts)^3 + 3(Ts)^2 + 3(Ts) + 1}, \tag{1}$$

where s is the Laplace operator, and equals $j\omega$ for plotting purposes;

$$T = \frac{1}{\omega_n}.$$

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[1] C. L. Longmire, "An *R-C* circuit giving over unity gain," *Tele-Tech*, vol. 6, pp. 40–41; April, 1947.

[2] H. M. James, N. B. Nichols, and R. S. Phillips, "Theory of Servomechanisms," p. 169 ff., McGraw-Hill Book Co., New York, N. Y.; 1947.

II. Development of Synthesis Procedure

Fig. 2 shows the two possibilities in the output connections with a specific input connection for three-terminal networks. Output$_a$ at terminals 2–2′ is the

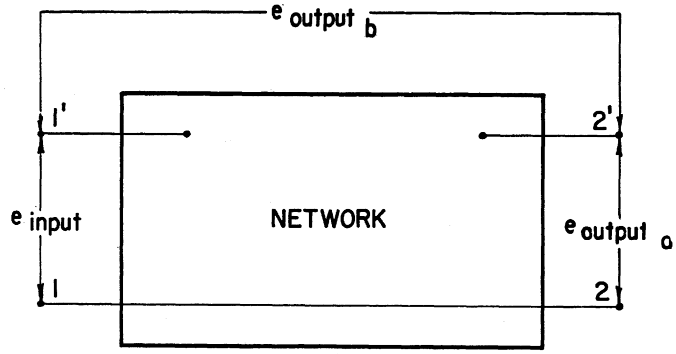

Fig. 2—General three-terminal network.

usual output connection. In this case, the common terminal 1–2 is ground, with the voltage transfer ratio given by

$$\frac{e_{\text{out}a}}{e_{\text{in}}} = \frac{f(s)}{F(s)}, \tag{2}$$

where

$f(s)$ = polynomial in s

$F(s)$ = polynomial in s of equal or higher degree than the numerator.

For resistor-capacitor networks in general, the roots of the denominator must be simple, real, and negative. In certain cases, a good approximation of the transfer function can be made with multiple poles in the denominator. This is equivalent, for instance, to an assumption that the sections in a ladder network do not load one another. The roots of the numerator, however, can lie anywhere in the complex plane. In other words, nothing can be specified about the location of the zeroes of the transfer function of a resistor-capacitor network in general.

The one other possibility of taking the output from the network of Fig. 2 is indicated by output$_b$ across terminals 1′–2′. Assume that output$_b$ is connected to an infinite load such as the grid of a vacuum tube. The common lead 1′ may be grounded. This type of connection changes the entire character of the network. Output$_b$ is now, in effect, the difference between the input and output$_a$;

$$e_{\text{out}b} = e_{\text{in}} - e_{\text{out}a}. \tag{3}$$

If, for instance, a resistor-capacitor ladder network produces a phase shift of π radians at some frequency, output_a is now expressed as

$$e_{out_a} = -ke_{in}, \quad (4)$$

where $0<k<1$. Thus, if the 1′–1 terminals have polarity marks +, −, respectively, then the 2′–2 terminals have polarity marks −, +, and output_b is then greater than the input

$$e_{out_b} = (1+k)e_{in}. \quad (5)$$

Thus a gain greater than unity is obtained from a passive resistor-capacitor network without the use of amplifiers.

Generally, with (2) representing the transfer from input to output_a, output_b is then expressed by the following:

$$\frac{e_{out_b}}{e_{in}} = 1 - \frac{f(s)}{F(s)} \quad (6a)$$

$$= \frac{F(s) - f(s)}{F(s)}. \quad (6b)$$

It is convenient and now possible to obtain a synthesis procedure based upon known methods. Therefore, suppose a certain specified transfer function is to be synthesized and is given as the ratio of two polynomials

$$\frac{e_{out}}{e_{in}} = \frac{f_1(s)}{F(s)}. \quad (7)$$

This function is specified as having unity gain at zero frequency and a gain greater than unity in some band. It is desired to synthesize this transfer ratio with as little, if any, additional amplification as is possible. From (6b) and (7), using output_b in Fig. 2,

$$\frac{e_{out_b}}{e_{in}} = \frac{f_1(s)}{F(s)} = \frac{F(s) - f(s)}{F(s)} \quad (8a)$$

or

$$f(s) = F(s) - f_1(s). \quad (8b)$$

This equation states that $f(s)$ is the numerator and $F(s)$ is the denominator of a transfer function given in (2), which is to be synthesized using known methods; for instance, the Bower-Ordung synthesis procedure.[3] The numerator of the function to be synthesized is the difference between the denominator and numerator of the specified transfer function, and the denominator is the same as the specified transfer function. The passive resistor-capacitor network then used in connection b of Fig. 2 satisfies the specified transfer function, gives gains greater than unity, and requires less over-all additional amplification.

The method involves taking the specified transfer function, and according to (8), changing it to another transfer ratio which is synthesized by usual means. It is evident that the derived function which is actually synthesized is never more complicated and is often simpler in form than the specified transfer function. Thus, not only is there a saving in amplification, but possibly also in the number of components.

III. Gain Considerations

The theoretical maximum voltage gain obtainable from a general type of RC ladder network is two. Each branch of the general ladder may consist of a parallel combination of a resistor and condenser. If the network consists of n sections, $n>2$, with all the sections having the same resistor-capacitor product, and if it is assumed that each succeeding section does not load the previous section, a vector diagram yields the following as the gain at some frequency:

$$\frac{e_{out}}{e_{in}} = 1 + \cos^n\left(\frac{\pi}{n}\right) \qquad n > 2 \quad (9a)$$

where the total phase shift due to the network is π radians. A phase shift of π radians can be shown to yield maximum gain. Since the cosine has a maximum value of unity at $\pi/n=0$ and has some value between 0 and 1 for other values of π/n, and since a fraction raised to an integer power n is less than the original value, then

$$\frac{e_{out}}{e_{in}} \leqq 2. \quad (9b)$$

For a comparison to this maximum, the theoretical maximum gain of a specific four section ladder shown in Fig. 3 is 1.25, assuming no loading and $\pi/4$ phase shift

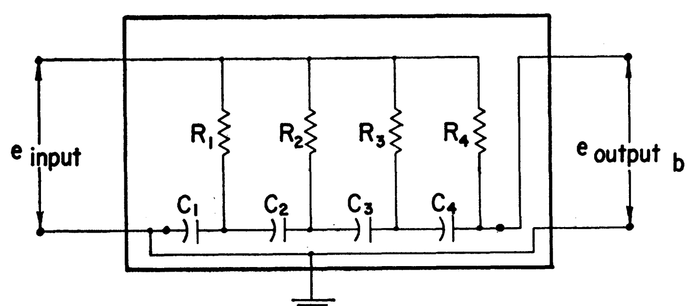

Fig. 3—Four-section ladder network.

from each section. On the other hand, the same type of circuit, made up of four identical sections, having the same impedance as the first section in the previous case has a gain magnitude of only 1.02 due to loading effects. This is calculated on the basis of zero source impedance and infinite load impedance. This comparison figure is arrived at by simply calculating the ratio of the output over the input voltage from the transfer determinant for a four-mesh ladder.

It has been shown that it is possible to obtain a saving in gain by a factor of two in designing so-called "lead" networks. It should be pointed out that a much greater

[3] J. L. Bower and P. F. Ordung, "The synthesis of resistor-capacitor networks," Proc. I.R.E., vol. 38, pp. 263–269; March, 1950.

saving in gain is possible in the synthesis of so-called "lag" networks having unity gain at zero frequency.

IV. Examples

As the first example, let the specified transfer function be that stated in (1) and shown graphically in Fig. 1. After the synthesis procedure is carried out, the result is a three-stage *RC* network of the type shown in Fig. 3, with all the *RC* products being made equal. The impedance level of each section should be raised by about a factor of ten. Note that this network yields complex zeroes which are impossible to obtain with the conventional connection of the ladder network. Furthermore, Fig. 3 indicates that the network may be nonminimum phase. A ladder of this sort, even in the connection shown, is still a minimum phase network as seen from an inspection of its transfer function (1).

The second example will show another simple case, often arising in servo synthesis, in which a passive resistor-capacitor network has unity gain at zero frequency and a gain greater than unity over a specified band.

Transfer function to be synthesized is given in (10).

$$\frac{e_{\text{out}b}}{e_{\text{in}}} = \frac{(1 + T_1 s)}{(1 + T_2 s)(1 + T_3 s)} = \frac{f_1(s)}{F(s)} \tag{10}$$

where $T_1 > T_2 > T_3$. Following the synthesis procedure, a new function is formed

$$\frac{f(s)}{F(s)} = \frac{F(s) - f_1(s)}{F(s)}$$

$$= \frac{s\left[1 + \dfrac{T_2T_3}{T_2 + T_3 - T_1} s\right](T_2 + T_3 - T_1)}{(1 + T_2 s)(1 + T_3 s)}. \tag{11}$$

In this case, a network is designed for $T_2 + T_3 > T_1$. With the condition imposed upon the T's in (10) it can be shown that

$$T_a = \frac{T_2T_3}{T_2 + T_3 - T_1} > T_1. \tag{12}$$

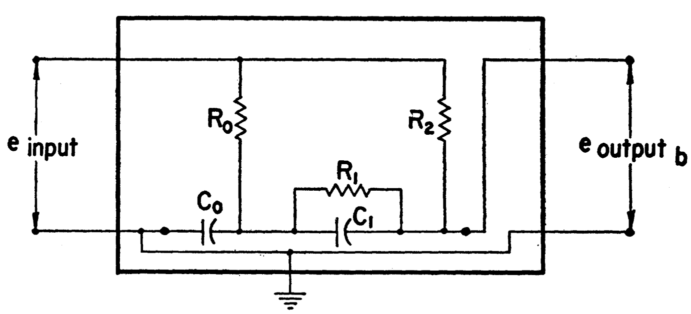

Fig. 4—Synthesized network satisfying (10).

$$T_3 = R_0C_0$$
$$T_a = R_1R_2C_1$$
$$T_2 = \frac{R_1R_2C_1}{R_1 + R}.$$

The usual synthesis procedure yields the result shown in Fig. 4, with the impedance level of the second section raised to minimize loading effects. An exact synthesis of (11) carried out according to the Bower-Ordung procedure yields the network shown in Fig. 5. The values

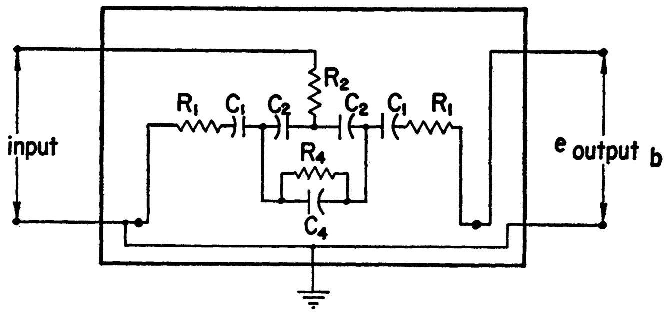

Fig. 5—Exact synthesis satisfying (10).

of the parameters in this network depend upon the specific values of T_a, T_2, and T_3 with the same restrictions on the T's as given in (10)

$$T_a > T_2 > T_3.$$

V. Suggestions for Further Work

While there are only two possible output connections in a three-terminal network, there are five such possibilities in a four-terminal network with only one of these connections usually used. These possibilities could be investigated in order to ascertain any advantages that may be derived from their use.

Although it seems intuitively evident that the maximum gain possible through the use of Fig. 2 and any network is two, a rigorous proof is not evident, particularly if loading is taken into account.

Perhaps the most important link missing in the synthesis procedure is a method which enables one to directly recognize a specified transfer function as being realizable by means of a particularly connected three- or four-terminal network. A general investigation of the different types of connections and their limitations in the two types of networks may point to a solution of this problem.

VI. Conclusions

It is possible to obtain a passive resistor-capacitor network having a gain greater than unity over a band of frequencies with unity gain at zero frequency. This is achieved very often with a saving in the number of components when compared to other methods. Furthermore, complex zeroes in the transfer ratio are easily obtainable even using ladder-type networks in a connection as shown in Fig. 2. The general resistor-capacitor network can also be made to perform as a tuned *LC* element by judicious choice of the zeroes in the transfer ratio. It is important to recognize the fact that the proper location of the zeroes of a transfer function can aid materially in synthesis problems.